

# Unfurling Khovanov-Lauda-Rouquier algebras

BEN WEBSTER<sup>1</sup>

Department of Mathematics

University of Virginia

Charlottesville, VA

Email: bwebster@virginia.edu

*Abstract.* In this paper, we study the behavior of categorical actions of a Lie algebra  $\mathfrak{g}$  under the deformation of their spectra. We give conditions under which the general point of a family of categorical actions of  $\mathfrak{g}$  carry an action of a larger Lie algebra  $\tilde{\mathfrak{g}}$ , which we call an **unfurling** of  $\mathfrak{g}$ . This is closely related to the folding of Dynkin diagrams, but to avoid confusion, we think it is better to use a different term.

Our motivation for studying this topic is the difficulty of proving that explicitly presented algebras and categories in the theory of higher representation theory have the “expected size.” Deformation is a powerful technique for showing this because of the upper semicontinuity of dimension under deformation. In particular, we’ll use this to show the non-degeneracy (in the sense of Khovanov-Lauda) of the 2-quantum group  $\mathcal{U}$  for an arbitrary Cartan datum and any homogeneous choice of parameters.

## 1. INTRODUCTION

The categorification of Lie algebras and their representations has proven to be a rich and durable subject since its introduction roughly a decade ago. This theory produces a 2-category  $\mathcal{U}$  depending on the choice of a Cartan datum and a choice of parameters; a representation of this 2-category is called a **categorical Lie algebra action** of the Kac-Moody algebra  $\mathfrak{g}$  corresponding to the Cartan datum. Many interesting categories carry a categorical Lie algebra action, though it must be admitted that most of the interesting examples are for a Cartan datum of (affine) type A.

However, since the 2-quantum group  $\mathcal{U}$  was first defined by Khovanov-Lauda [KL10] and Rouquier [Roua], it has been haunted by a serious problem: since it is presented by generators and relations, it is hard to check that it is not smaller than expected. Specifically, in [KL10], it’s proven that there is a surjective map from the modified quantum group  $\dot{\mathcal{U}}$  to the Grothendieck group of  $\mathcal{U}$  and that the dimension of 2-morphism spaces between 1-morphisms in  $\mathcal{U}$  is bounded above by a variation on Lusztig’s bilinear form on the modified quantum group  $\dot{\mathcal{U}}$ . If equality holds in this bound, then  $\dot{\mathcal{U}}$  is isomorphic to the Grothendieck group and we call the corresponding categorification **non-degenerate**.

As a general rule, proving non-degeneracy depends on constructing appropriate representations where one can show that no unexpected relations exist between 2-morphisms in  $\mathcal{U}$ . This is done for  $\mathfrak{sl}_n$  in [KL10], using an action on the cohomology

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of flag varieties. The next major step was independent proofs by Kang-Kashiwara [KK12] and the author [Web13] that the simple highest weight representations of  $\mathfrak{g}$  possess categorifications which are non-degenerate in an appropriate sense; this is sufficient to prove the non-degeneracy for finite type Cartan data.

This allowed significant progress, but along with many other techniques (such as connections to quiver varieties studied in [CKL13, Roub, Webb]), it has an unfortunate defect. Recall that the *open Tits cone* of a Cartan datum is the elements in the orbit of a dominant weight under the Weyl group; for example, if  $\mathfrak{g}$  is affine, then the open Tits cone is the set of weights which are not non-zero multiples of  $\delta$ . This set is convex and every weight of a highest or lowest weight integrable representation lies inside the open Tits cone. If the Cartan datum is of infinite type, then no information about a weight  $\lambda$  outside the open Tits cone is contained in any of these representations, since the corresponding idempotent  $1_\lambda \in \dot{\mathcal{U}}$  kills any integrable highest or lowest weight representation.

Thus, if we are to understand non-degeneracy for weights outside the open Tits cone, we must have access to representations which are not highest or lowest weight. For our purposes, the most promising are those given by a tensor product of highest and lowest weight representations (perhaps many of each type). A construction of such categorifications was given in [Web15], but the non-degeneracy proof given there is only valid for weights inside the Tits cone. Thus, to access these other weights, we must give a new argument for the non-degeneracy of these tensor product categorifications, which will then imply the non-degeneracy of  $\mathcal{U}$ . In particular, we prove that:

**Theorem A** (Theorem 4.10) *Fix a field  $\mathbb{k}$  and consider any Cartan datum  $(I, \langle -, - \rangle)$ , and choice of the polynomials  $Q_{ij}(u, v) \in \mathbb{k}[u, v]$  which is homogeneous (in the sense discussed in Section 2.1). The associated 2-quantum group  $\mathcal{U}$  is non-degenerate, and the Grothendieck group of  $\mathcal{U}$  is  $\dot{\mathcal{U}}$ .*

While an interesting theorem in its own right, the techniques introduced here to prove this result have a significance of their own. A simple, but underappreciated, technique for proving these sort of non-degeneracy arguments is the upper semicontinuity of dimension under deformation. Perhaps calling this “underappreciated” is unfair, since it is certainly a well-known and much used trick, but at least this author wishes he had exploited it more systematically earlier.

Thus, much of this paper will be dedicated to an exploration of the behavior of categorical actions under deformation. Let  $R$  be the KLR algebra of the Cartan datum  $(I, \langle -, - \rangle)$  (defined in Section 2.1). We wish to consider quotients of this algebra where the dots (the elements usually denoted  $y_k \in R$ ) have a fixed spectrum; of course, all of these quotients can be packaged together into a completion  $\hat{R}$ . Most often, people have studied representations where the elements  $y_k$  act nilpotently (all gradeable

finite dimensional representations have this property), but we can also have them act with certain fixed non-zero eigenvalues. Given a choice of spectrum for the dots, we have an associated graph with vertex set  $\tilde{I}$ , with its associated Cartan datum. There's a natural map  $\tilde{I} \rightarrow I$ , which one can informally think of as a “branched cover” of the Cartan datum  $(I, \langle -, - \rangle)$ .

This is closely related to the phenomenon of **folding** of Dynkin diagrams, but due to some technical differences, we think it would be misleading to use the term “folding” here. Thus, we call  $\tilde{I}$  an **unfurling** of  $I$  (and  $I$  a **furling** of  $\tilde{I}$ ). Note that whereas  $I$  is not necessarily symmetric as a Cartan datum (i.e. it has roots of different lengths),  $\tilde{I}$  is symmetric. To give the reader a sense of this operation, let us discuss some examples:

- If  $I$  is simply-laced, then  $\tilde{I}$  will be a topological cover of  $I$ , such as an  $A_\infty$  graph covering an  $n$ -cycle, or the trivial cover  $\tilde{I} \cong I \times U$ .
- If  $I$  is not simply laced, we can arrange for  $\tilde{I}$  to be given by a simply-laced Cartan datum with an isomorphism  $\mathfrak{g} \cong \tilde{\mathfrak{g}}^\sigma$  for some diagram automorphism  $\sigma$ . Note, this means that  $I$  is the Langlands dual of what is usually called a folding of  $\tilde{I}$  for the automorphism  $\sigma$ .

At the moment, it is unclear to the author what, if any, is the relationship between this work and that of McNamara [McN] and Elias [Eli] which also combine the ideas of categorification and folding. Obviously, this would be an interesting topic for future consideration.

We always have a map of Lie algebras  $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ , and this map has a categorical analogue:

**Theorem B** (Proposition 3.3, Theorem 3.4)

- (1) *The completion  $\hat{R}$  is isomorphic to the completion of the KLR algebra  $R$  for  $\tilde{I}$  with respect to its grading.*
- (2) *A categorical action of  $\mathfrak{g}$  satisfying certain spectral conditions corresponding to  $\tilde{I}$  also carries an action of  $\tilde{\mathfrak{g}}$ .*

This theorem is particularly useful when we have a family of categorical  $\mathfrak{g}$ -modules. We will typically have a categorical action of a larger Lie algebra at the generic point of the deformation, but this action will fail to exist at certain special points where the spectrum drops in size. Explicitly, the operators for  $\tilde{\mathfrak{g}}$  are defined as the images of idempotents acting on 1-morphisms in  $\mathcal{U}$ ; the formula for these idempotent depends on dividing by certain expressions that the spectral conditions guarantee are invertible. At points where these denominators vanish, the idempotents and thus their images may no longer be well-defined.

That is, we can take a categorical  $\mathfrak{g}$ -module  $C$  which does not have a  $\tilde{\mathfrak{g}}$ -action, find a deformation  $\tilde{C}$  where the generic point satisfies the spectral conditions, and then

exploit the categorical  $\tilde{g}$ -action to prove the original categorification  $C$  is not smaller than expected.

This is our strategy for studying the categorifications  $\mathcal{X}^\lambda$  of tensor products of highest and lowest weight representations. These live in a natural family  $\mathcal{X}^\lambda$ , where the generic point  $\mathcal{X}_{\mathbb{K}}^\lambda$  categorifies an irreducible representation of a larger Lie algebra. We can then apply the categorification result for this larger irreducible (from [KK12, Weba]) to prove the non-degeneracy of  $\mathcal{X}^\lambda$  and  $\mathcal{U}$ .

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## 2. BACKGROUND

**2.1. The KLR algebra.** Throughout, we'll fix a finite set  $I$ , and a Cartan datum on this set. We'll consider the root lattice  $X$ , the free abelian group generated by the simple roots  $\alpha_i$  for  $i \in I$ , and we let  $\langle -, - \rangle$  denote the symmetric bilinear form on this abelian group attached to the Cartan datum.

The coroot lattice  $X^\vee$  is the free abelian group generated by the symbols  $\alpha_i^\vee$ . We have a pairing of  $X^\vee \times X \rightarrow \mathbb{Z}$  such that  $\alpha_i^\vee(\alpha_j) := \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$ . The matrix of this form is the Cartan matrix  $C = (c_{ij} = \alpha_i^\vee(\alpha_j))$ . Note that  $d_i = \langle \alpha_i, \alpha_i \rangle / 2$  are symmetrizing coefficients for this Cartan matrix:  $d_i c_{ij} = d_j c_{ji} = \langle \alpha_i, \alpha_j \rangle$  for all  $i, j$ . For our purposes a **weight** is an element of the dual lattice to  $X^\vee$ , so the pairing above lets us consider each element of the root lattice as a weight<sup>2</sup>. Given a weight  $\lambda$ , we let  $\lambda^i = \alpha_i^\vee(\lambda)$ .

Fix an algebraically closed field  $\mathbb{k}$  of characteristic coprime to all  $d_i$ , and choose polynomials  $P_{ij}(u, v)$  such that the product  $Q_{ij}(u, v) = P_{ij}(u, v)P_{ji}(v, u)$  is homogeneous of degree  $-\langle \alpha_i, \alpha_j \rangle = -2d_i c_{ij} = -2d_j c_{ji}$  when  $u$  is given degree  $2d_i$  and  $v$  degree  $2d_j$ . Let  $p_{ij} = P_{ij}(1, 0)$ . Let  $g_{ij} = \gcd(-c_{ij}, -c_{ji})$  and  $h_{ij} = -c_{ij}/g_{ij}$ . We must have that

$$P_{ij}(u, v) = p_{ij} \prod_{a_{ij}^{(k)} \in A_{ij}} (u^{1/d_i} - a_{ij}^{(k)} v^{1/d_j})$$

where  $A_{ij} = \{a_{ij}^{(k)}\}$  is the multiset of roots of  $P_{ij}(x^{d_i}, 1)$ , considered with multiplicity. Let  $B_{ij} = \{b_{ij}^{(k)} = (a_{ji}^{(k)})^{-1}\}$  be the reciprocals of these numbers. Note that

$$Q_{ij}(u, v) = t_{ij} \prod_{a_{ij}^{(k)} \in A_{ij}} (u^{1/d_i} - a_{ij}^{(k)} v^{1/d_j}) \prod_{b_{ij}^{(k)} \in B_{ij}} (u^{1/d_i} - b_{ij}^{(k)} v^{1/d_j})$$

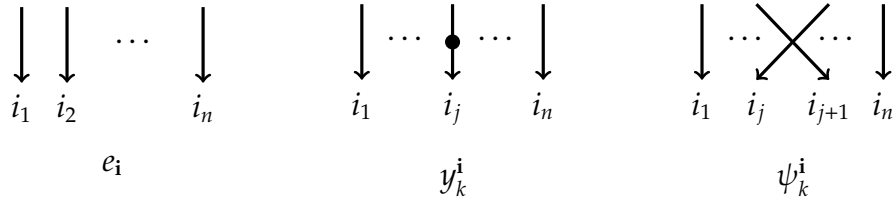
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<sup>2</sup>In terms of Kac-Moody groups, these are weights of the torus of the derived subgroup of a Kac-Moody group of this type; the rest of the torus will not play an important role for us.

Homogeneity requires that  $P_{ij}(x^{1/h_{ij}}, 1)$  be a polynomial. We'll also let  $\mathcal{A}_{ij} = \{\alpha_{ij}^{(k)}\}$  be the roots of  $P_{ij}(x^{1/h_{ij}}, 1)$ , again considered with multiplicity; note that the elements of  $\mathcal{A}_{ij}$  are the  $d_i h_{ij}$ th roots of the elements of  $\mathcal{A}_{ji}$ . Furthermore, we have that  $d_i h_{ij} = d_j h_{ji} = \text{lcm}(d_i, d_j)$ . We also let  $\mathcal{B}_{ij} = \{\beta_{ij}^{(k)} = (\alpha_{ji}^{(k)})^{-1}\}$ . As before, the elements of  $\mathcal{B}_{ij}$  are the  $d_i h_{ij}$ th roots of the elements of  $\mathcal{B}_{ij}$ .

**Definition 2.1** Let  $R$  denote the KLR algebra with generators given by:

- The idempotent  $e_i$  which is straight lines labeled with  $(i_1, \dots, i_n) \in I^n$ .
- The element  $y_k^i$  which is just straight lines with a dot on the  $k$ th strand.
- The element  $\psi_k^i$  which is a crossing of the  $i$  and  $i+1$ st strand.



and relations:

(2.1a) unless  $i = j$

(2.1b) unless  $i = j$

(2.1c) +

(2.1d) +

$$(2.1e) \quad \begin{array}{c} \text{Diagram: two strands crossing, both labeled } i, \text{ with arrows pointing towards the crossing} \\ \text{Diagram: two strands crossing, labeled } i \text{ and } j, \text{ with arrows pointing towards the crossing} \end{array} = 0 \quad \text{and} \quad \begin{array}{c} \text{Diagram: two strands crossing, labeled } i \text{ and } j, \text{ with arrows pointing towards the crossing} \\ \text{Diagram: a box labeled } Q_{ij}(y_1, y_2) \text{ with four vertical lines (two on each side) connecting to the strands} \end{array}$$

$$(2.1f) \quad \begin{array}{c} \text{Diagram: three strands labeled } i, j, k \text{ with arrows pointing towards the crossing} \\ \text{Diagram: three strands labeled } i, j, k \text{ with arrows pointing away from the crossing} \end{array} = \begin{array}{c} \text{Diagram: three strands labeled } i, j, k \text{ with arrows pointing towards the crossing} \\ \text{Diagram: three strands labeled } i, j, k \text{ with arrows pointing away from the crossing} \end{array} \quad \text{unless } i = k \neq j$$

$$(2.1g) \quad \begin{array}{c} \text{Diagram: three strands labeled } i, j, i \text{ with arrows pointing towards the crossing} \\ \text{Diagram: three strands labeled } i, j, i \text{ with arrows pointing away from the crossing} \end{array} = \begin{array}{c} \text{Diagram: three strands labeled } i, j, i \text{ with arrows pointing towards the crossing} \\ \text{Diagram: three strands labeled } i, j, i \text{ with arrows pointing away from the crossing} \end{array} + \begin{array}{c} \text{Diagram: a box labeled } \frac{Q_{ij}(y_3, y_2) - Q_{ij}(y_1, y_2)}{y_3 - y_1} \text{ with four vertical lines (two on each side) connecting to the strands} \end{array}$$

Fix a countable set  $U_i \subset \mathbb{k} \setminus \{0\}$  for each  $i \in I$ ; for each  $u \in U_i$ , we fix a choice of  $d_i$ th root, which we denote  $u^{1/d_i}$  (there are  $d_i$  choices, since  $d_i$  is coprime to the characteristic of  $\mathbb{k}$ ). Since  $U_i$  is countable, we can write it as a union of nested finite sets  $U_i^{(N)}$  for  $N \in \mathbb{Z}_{\geq 0}$  (this choice is purely for technical reasons, and nothing we do will depend on it).

**Definition 2.2** Let  $\hat{R}_n$  be the completion of the KLR algebra  $R_n$  by the system of ideals  $\mathcal{I}_N$  generated by  $e_i \prod_{u \in U_{i_j}^{(N)}} (y_j - u)^N$  for all  $j \in [1, n]$  and  $\mathbf{i} \in I^n$ .

This is the coarsest completion where we require that the topological spectrum of a dot on a strand with label  $i$  lives in the set  $U_i$ .

**2.2. Valued graphs.** We'll follow the conventions of Lemay [Lem] in this section. For simplicity, "graph" will always mean a graph without loops.

**Definition 2.3** A **relatively valued graph** is an oriented graph with vertex set  $I$  with a pair of rational numbers  $(\eta_e, \nu_e)$  assigned to each edge such that there exist  $d_i \in \mathbb{Q}$  for each  $i \in I$  such that  $d_i \eta_e = d_j \nu_e$  for  $e: i \rightarrow j$ .

An **absolutely valued graph** is an oriented graph as above with a choice of rational numbers  $d_i$  for each vertex  $i$  and  $m_e$  for each edge  $e$ .

Each absolutely valued graph has an associated relatively valued graph with

$$\eta_e = \frac{m_e}{d_i} \quad \nu_e = \frac{m_e}{d_j}$$

for an edge  $e: i \rightarrow j$ , and every relatively valued graph has this form. The values  $\eta_e$  and  $v_e$  will be integers if  $d_i$  and  $m_e$  are and  $\text{lcm}(d_i, d_j)$  divides  $m_e$  for an edge  $e: i \rightarrow j$ . Note that relatively valued graphs have a natural notion of **Langlands duality**, given by switching  $\eta_e$  and  $v_e$ . We attach a Cartan matrix to each such graph without loops, with  $c_{ii} = 2$  and

$$c_{ij} = - \sum_{e: i \rightarrow j} \eta_e - \sum_{e: j \rightarrow i} v_e = -\frac{1}{d_i} \sum_{e: i \rightarrow j} m_e - \frac{1}{d_j} \sum_{e: j \rightarrow i} m_e.$$

Note that Langlands duality transposes this Cartan matrix.

Having chosen  $P_{ij}(u, v)$  for each pair  $i, j \in I^2$ , we can canonically associate an absolutely valued graph with vertex set  $I$  where we add an edge  $i \rightarrow j$  whenever  $P_{ij}(u, v)$  is non-constant. The values  $d_i$  are as before, and  $m_e = \deg P_{ij}(x^{d_i}, 0) = \deg P_{ij}(0, x^{d_j})$ . In the associated relatively valued quiver, the values we add to this edge are  $(\deg P_{ij}(x, 0), \deg P_{ij}(0, x))$ . The Cartan matrix of the result is our original Cartan matrix  $C$ .

Given a graph homomorphism between two valued graphs, we can consider various forms of compatibility between the valuations on the two graphs. One notion considered by Lemay [Lem] is a morphism of valued graphs: this is a homomorphism of graphs where the appropriate statistics  $(\eta_*, v_*, m_*, d_*)$  are preserved; this is too inflexible for our purposes. Instead, we'll consider a set of maps which are more analogous to topological covers.

**Definition 2.4** We call a map  $f: X \rightarrow Y$  of relatively valued graphs a **furling** if given any  $y, y' \in Y$ , and  $x \in f^{-1}(y)$ , we have that each edge  $d: y \rightarrow y'$  and each edge  $e: y' \rightarrow y$ ,

$$v_d = \sum_{x' \in f^{-1}(y')} \sum_{\substack{d': x \rightarrow x' \\ f(d')=d}} v_{d'} \quad \eta_e = \sum_{x' \in f^{-1}(y')} \sum_{\substack{e': x' \rightarrow x \\ f(e')=e}} \eta_{e'}.$$

The notion of a furling is very closely related to a “folding,” but we won't use this term, since it usually applies to the Langlands dual of the operation above, and implies the existence of a group action. We'll call  $Y$  a **furling** of  $X$  and  $X$  an **unfurling** of  $Y$ . A morphism of relatively valued graphs which is also a topological cover is a furling.

Note that if  $X$  has an compatible absolute valued structure such that  $d_x$  and  $m_e$  is constant on the fibers of  $f$  and these fibers are finite, then for an edge  $e: y' \rightarrow y$ , we have that

$$\eta_e = \sum_{x' \in f^{-1}(y')} \frac{1}{d_{x'}} \sum_{\substack{e': x' \rightarrow x \\ f(e')=e}} m_{e'} \quad v_e = \sum_{x \in f^{-1}(y)} \frac{1}{d_x} \sum_{\substack{e': x' \rightarrow x \\ f(e')=e}} m_{e'}.$$

Thus, we can choose an absolute valued structure on  $Y$  such that

$$d_y = \frac{d_x}{|f^{-1}(y)|} \quad m_e = \frac{\sum_{f(e')=e} m_{e'}}{|f^{-1}(y)| \cdot |f^{-1}(y')|}.$$

As defined here,  $d_y$  and  $m_e$  may not be integers, but if  $Y$  is finite, then we can always just multiply every  $d_y$  and  $m_e$  by  $\text{lcm}(|f^{-1}(y)|)$  to clear denominators.

One special case of particular interest is when  $X$  is given the trivial valuation  $d_x = m_e = v_e = \eta_e = 1$  and is equipped with an admissible automorphism  $\sigma$ ; recall that we call an automorphism of a graph **admissible** if no edges connect two vertices in the same orbit under the action. We let  $Y$  be the quotient graph  $X/\sigma$  and  $f: X \rightarrow Y$  the obvious projection map. In this case, we have

$$(2.2) \quad d_y = \frac{1}{|f^{-1}(y)|} \quad m_e = \frac{|f^{-1}(e)|}{|f^{-1}(y)| \cdot |f^{-1}(y')|}.$$

This is the Langlands dual of the “folding” discussed in [Lem, §1] (which is the more common way of associating a Cartan matrix to a graph with automorphism).

**Lemma 2.5** *Given a furling  $f: X \rightarrow Y$ , for any fixed  $y, y' \in Y$  and  $x' \in f^{-1}(y')$  we have that:*

$$c_{yy'} = \sum_{x \in f^{-1}(y)} c_{xx'}.$$

*Proof.*

$$\begin{aligned} c_{yy'} &= - \sum_{e: y \rightarrow y'} \eta_e - \sum_{e: y' \rightarrow y} v_e \\ &= - \sum_{x \in f^{-1}(y)} \left( \sum_{e: x \rightarrow x'} \eta_e - \sum_{e: x' \rightarrow x} v_e \right) \\ &= \sum_{x \in f^{-1}(y)} c_{xx'} \quad \square \end{aligned}$$

We assume from now on for all valued graphs appearing that the matrix  $C$  is a generalized Cartan matrix (in particular, all off-diagonal entries are negative integers).

**Definition 2.6** *Given a valued graph  $X$ , let  $\mathfrak{g}_X$  be the associated derived Kac-Moody algebra generated by  $E_i, F_i, H_i$  with the relations*

$$(2.3) \quad [H_i, E_j] = c_{ij} E_j \quad [H_i, F_j] = -c_{ij} F_j \quad [E_i, F_j] = \delta_{ij} H_i$$

$$(2.4) \quad \text{ad}_{E_i}^{1-c_{ij}} E_j = \text{ad}_{F_i}^{1-c_{ij}} F_j = 0.$$

A straightforward extension of [Kac90, 7.9] shows that:



**Proposition 2.7** *If  $f: X \rightarrow Y$  is a furling of valued graphs, there is an induced homomorphism of Kac-Moody algebras  $\mathfrak{g}_Y \rightarrow \mathfrak{g}_X$  given by the formulas:*

$$F_y \mapsto \sum_{x \in f^{-1}(y)} F_x \quad E_y \mapsto \sum_{x \in f^{-1}(y)} E_x \quad H_y \mapsto \sum_{x \in f^{-1}(y)} H_x.$$

*Proof.* The relations (2.3) are straightforward computations using Lemma 2.5. We have that:

$$\begin{aligned} \left[ \sum_{x \in f^{-1}(y)} H_x, \sum_{x' \in f^{-1}(y')} E_{x'} \right] &= \sum_{x' \in f^{-1}(y')} \left( \sum_{x \in f^{-1}(y)} c_{xx'} \right) E_{x'} = \sum_{x' \in f^{-1}(y')} c_{yy'} E_{x'} \\ \left[ \sum_{x \in f^{-1}(y)} H_x, \sum_{x' \in f^{-1}(y')} F_{x'} \right] &= \sum_{x' \in f^{-1}(y')} \left( \sum_{x \in f^{-1}(y)} -c_{xx'} \right) F_{x'} = \sum_{x' \in f^{-1}(y')} -c_{yy'} F_{x'} \\ \left[ \sum_{x \in f^{-1}(y)} E_x, \sum_{x' \in f^{-1}(y')} F_{x'} \right] &= \delta_{y,y'} \sum_{x \in f^{-1}(y)} H_x \end{aligned}$$

The remaining relations (2.4) follow from the Gabber-Kac theorem [GK81].  $\square$

**Definition 2.8** *Let  $\tilde{I}$  be the oriented graph whose vertex set is the pairs  $\{(i, u) \in I \times \mathbb{k} \mid u \in U_i\}$  with the edges oriented from  $(i, u)$  to  $(i', u')$  in bijection with the elements of  $A_{ij}$  such that  $u^{1/d_i} = a_{ii'}^{(k)}(u')^{1/d_{i'}}$ . We will consider this as an absolutely/relatively valued graph with trivial valuation  $\eta_e = v_e = d_i = m_e = 1$ .*

These edges are also in bijection with solutions to  $u^{h_{ij}} = \alpha_{ii'}^{(k)}(u')^{h_{ij}}$  for  $\alpha_{ii'}^{(k)} \in \mathcal{A}_{ii'}$  (which tend to be slightly easier to count).

**Definition 2.9** *We call a choice of spectra  $U_i$  **complete** if whenever  $Q_{ij}(u, u') = 0$  for  $u \in U_i$ , then  $u' \in U_j$ .*

Note that if all  $a_{ij}^{(k)}$ 's are  $m$ th roots of unity for some  $m$ , then any finite choice of  $U_i$ 's can be made complete by adding finitely many elements (for example, all products of elements of  $U_i$  and  $m$ th roots of unity will suffice). Any countable choice of spectra can be completed to a countable complete choice of spectra; we let  $V_i^{(0)} = U_i$  and define  $V_i^{(k)} = \{v \in \mathbb{k} \mid Q_{ij}(v, v') = 0 \text{ for } v' \in V_j^{(k-1)}\}$ . The union  $V_i^{(\infty)} = \cup_{i=0}^{\infty} V_i^{(k)}$  is a complete choice of spectra.

**Proposition 2.10** *If  $U_i$  is a complete choice of spectra, then the map  $\tilde{I} \rightarrow I$  is a furling of valued graphs.*

*Proof.* The unique edge  $e: i \rightarrow i'$  in  $I$  has preimages corresponding to each element  $u \in U_i$  and each root of  $P_{ii'}(u, x)$  as a polynomial in  $x$ . Thus the number of preimages  $e'$  is the degree of this polynomial in  $x$ , and each has  $v_{e'} = 1$ , so this agrees with

$v_e = \deg P_{ij}(0, x)$ . Similarly, if we consider the edge  $d: i' \rightarrow i$ , the edges are in bijection with the roots of  $P_{i'i}(x, u)$ , and  $\eta_d = \deg P_{i'i}(x, 0)$ . This completes the proof.  $\square$

The most important example is the so-called “geometric” parameters for the symmetric Cartan matrix for an oriented graph, where  $P_{i'i'}(u, v) = (u - v)^{\#i \rightarrow i'}$ . In this case,  $a_{i'i'}^{(k)} = 1$  for all  $k$  and  $(i, u)$  is connected to  $(i', u')$  by the same number of edges as  $i$  and  $i'$  if  $u = u'$  and none otherwise. Thus, if we choose  $U_i = U$  for some fixed set  $U \subset \mathbb{k}$ , this is a complete choice of spectra and  $\tilde{I} = I \times U$  with the obvious graph structure.

If  $I$  is simply laced, but not simply-connected, then we can obtain non-trivial covers as  $\tilde{I}$ . For example, if  $I$  is an  $n$ -cycle with its vertex set identified with  $\mathbb{Z}/n\mathbb{Z}$  with edges  $i \rightarrow i + 1$ . Fix some  $q \in \mathbb{k}$  and choose  $Q_{i,i+1}(u, v) = qu - v$ . If we fix  $U_0 \subset \mathbb{k}$  to be any subset closed under multiplication by  $q^n$ , then we have a complete choice of spectra with  $U_i = q^i U_0$ . The components of the graph  $\tilde{I}$  correspond to the orbits of multiplication by  $q$ ; these will be cycles if  $q$  is a root of unity, or  $A_\infty$  graphs if  $q$  is not.

On the other hand, if we have a nonsymmetric Cartan matrix, then we may find a more interesting result. If  $Q_{12}(u, v) = u^2 - v$  and  $d_1 = 1, d_2 = 2$  (so  $\mathfrak{g} = \mathfrak{sp}_4$ ), then the factorization shown earlier is that  $Q_{12}(u, v) = (u + v^{1/2})(u - v^{1/2})$ , and so  $a_{12}^{(1)} = 1$  and  $a_{12}^{(2)} = -1$ . Thus, the number of edges joining  $(1, x)$  to  $(2, y)$  is given by the number of solutions to  $x = \pm \sqrt{y}$ . Thus, every component of  $\tilde{I}$  is a subgraph of an  $A_3$  formed by  $(1, x) \rightarrow (2, x^2) \leftarrow (1, -x)$  (assuming  $1 \neq -1$ ).

More generally, let  $d = \text{lcm}(d_i)_{i \in I}$ . Let  $U_i$  be the  $d/d_i$ th roots of unity; there are  $d/d_i$  distinct roots of unity since  $d$  is coprime to the characteristic of  $\mathbb{k}$ . Assume that each  $a_{ij}^{(k)}$  is a  $d$ th root of unity for all  $i, j, k$ . For example, we can assume that

$$(2.5) \quad Q_{ij}(u, v) = \pm(u^{h_{ij}} - v^{h_{ji}})^{g_{ij}}$$

in which case,  $Q_{ij}(x^{d_i}, 1) = \pm(x^{d_i h_{ij}} - 1)^{g_{ij}}$ , so the multiset of  $a_{ij}^{(k)}$  and  $b_{ij}^{(k)}$  is given by the  $d_i h_{ij} = d_j h_{ji} = \text{lcm}(d_i, d_j)$  roots of unity each with multiplicity  $g_{ij}$ . Alternatively, we have  $\alpha_{ij}^{(k)} = \beta_{ij}^{(k)} = 1$  for any  $k$ .

We have that each  $u \in U_i$  is connected by  $c_{ji}$  edges to elements of  $U_j$ , given by the  $h_{ji}$ th roots of  $\alpha_{ij}^{(k)} u^{h_{ij}}$ . If  $Q_{ij}$  is as in (2.5), then for each  $d/d_i h_{ij}$ th root of unity  $\xi$ , we connect each  $h_{ij}$ th root of  $\xi$  in  $U_i$  to each  $h_{ji}$ th root of  $\xi$  in  $U_j$  with  $g_{ij}$  edges (with orientation depending on  $P_{ij}$ ). Let  $\zeta$  be a primitive  $d$ th root of unity.

**Proposition 2.11** *Assuming  $a_{ij}^{(k)}$  is a  $d$ th root of unity for all  $i, j, k$  and  $U_i$  is the  $d/d_i$ th roots of unity, the map  $\sigma: (i, u) \mapsto (i, \zeta^{d_i} u)$  is an admissible automorphism of the graph  $\tilde{I}$ ; the map  $\tilde{I} \rightarrow I = \tilde{I}/\sigma$  induces the relative valued structure on  $I$  associated to the polynomials  $P_{ij}$ .*

Note that the absolute weighting of (2.2) is the symmetrization we have chosen for our Cartan matrix divided by  $d$ , since  $|f^{-1}(i)| = d/d_i$ .

## 3. COMPLETED KLR ALGEBRAS

**3.1. An isomorphism of completed KLR algebras.** In this section we'll show how the completions of KLR algebras for  $I$  we discussed earlier are related to the KLR algebras of  $\tilde{I}$ .

Let  $\mathbf{j} = (j_1 = (i_1, u_1), \dots, j_n = (i_n, u_n))$ . By abstract Jordan decomposition, for any element  $x \in \hat{R}$  and  $u \in \mathbb{k}$ , there is unique idempotent  $e$  in the quotient  $\hat{R}/I_N$  whose image is the generalized  $u$ -eigenspace of  $x$  acting by left multiplication; by uniqueness, these are compatible under the quotient maps, and thus give an idempotent in  $\hat{R}$ . Performing this construction inductively, we can consider any commuting set of elements  $\{x_j\}$  and construct an idempotent projecting to their simultaneous generalized eigenspace in  $\hat{R}/I_N$  for any choice of scalars  $\{u_j\}$ .

**Definition 3.1** *Let  $\epsilon_j$  be the resulting idempotent where we perform this construction with  $e_i$  having eigenvalue 1 and  $y_j$  eigenvalue  $u_j$  for all  $j$ .*

In particular,  $e_i \epsilon_j = \epsilon_j e_i = \epsilon_j$ , and  $(y_j - u_j) \epsilon_j$  is topologically nilpotent.

There is a unique  $d_{i_k}$ th root of  $y_k \epsilon_j$  such that  $(y_k^{1/d_{i_k}} - u_k^{1/d_{i_k}}) \epsilon_j$  is topologically nilpotent. This is given by

$$y_k^{1/d_{i_k}} = u_k^{1/d_{i_k}} (u_k^{-1} y_k)^{1/d_{i_k}} = u_k^{1/d_{i_k}} \left( 1 + \frac{1}{d_{i_k}} (u_k^{-1} y_k - 1) + \binom{1/d_{i_k}}{2} (u_k^{-1} y_k - 1)^2 + \binom{1/d_{i_k}}{3} (u_k^{-1} y_k - 1)^3 + \dots \right).$$

Here the symbol  $\binom{1/d_{i_k}}{n}$  denotes the image of this binomial coefficient under the canonical map  $\mathbb{Z}[1/d_{i_k}] \rightarrow \mathbb{k}$ .

Consider the KLR algebra  $R$  of  $\tilde{I}$  with the symmetric Cartan datum associated to its graph structure: to avoid confusion, we'll denote the elements  $\psi_k, y_k, e_j$  of this algebra with sans serif letters. We use the geometric coefficients given by  $P_{jj'}(u, v) = (u - v)^{\#j \rightarrow j'}$ . We let  $A_k$  be the unique invertible element of the completion  $\hat{R}$  such that

$$A_k \epsilon_j = p_{i_k i_{k+1}} \epsilon_j \prod_{\substack{1/d_{i_k} \neq a_{i_k i_{k+1}}^{(m)} \\ u_k^{1/d_{i_k}} \neq a_{i_k i_{k+1}}^{(m)} u_{k+1}^{1/d_{i_{k+1}}}}} (y_k^{1/d_{i_k}} - a_{i_k i_{k+1}}^{(m)} y_{k+1}^{1/d_{i_{k+1}}}) \prod_{\substack{1/d_{i_k} = a_{i_k i_{k+1}}^{(m)} \\ u_k^{1/d_{i_k}} = a_{i_k i_{k+1}}^{(m)} u_{k+1}^{1/d_{i_{k+1}}}}} u_k^{1/d_{i_k}}$$

The multiplicative inverse  $A_k^{-1}$  makes sense because each non-scalar factor of  $A_k \epsilon_j$  is of the form  $(u_k^{1/d_{i_k}} - a_{i_k i_{k+1}}^{(m)} u_{k+1}^{1/d_{i_{k+1}}}) \epsilon_j$ , which is a non-zero multiple of the idempotent  $\epsilon_j$  plus a topologically nilpotent element. Note that:

$$(3.1) \quad P_{i_k i_{k+1}}(y_k, y_{k+1}) \epsilon_j = A_k \prod_{\substack{1/d_{i_k} = a_{i_k i_{k+1}}^{(m)} \\ u_k^{1/d_{i_k}} = a_{i_k i_{k+1}}^{(m)} u_{k+1}^{1/d_{i_{k+1}}}}} ((u_k^{-1} y_k)^{1/d_{i_k}} - (u_{k+1}^{-1} y_{k+1})^{1/d_{i_{k+1}}}) \epsilon_j$$

**Proposition 3.2** *There is a homomorphism  $v: R \rightarrow \hat{R}$  sending*

$$e_j \mapsto \epsilon_j \quad y_k e_j \mapsto ((u_k^{-1} y_k)^{1/d_{i_k}} - 1) \epsilon_j$$

$$\psi_k \mathbf{e}_j \mapsto \begin{cases} A_k^{-1} \psi_k \mathbf{e}_j & i_k \neq i_{k+1} \\ ((y_{k+1} - y_k) \psi_k + 1) \mathbf{e}_j & i_k = i_{k+1}, u_k \neq u_{k+1} \\ \psi_k \mathbf{e}_j & j_k = j_{k+1} \end{cases}$$

*Proof.* One can see that this is a homomorphism by comparing polynomial representations. By [Roua, 3.12], there is a polynomial representation of  $R_n$  on  $Z = \oplus_{\mathbf{i} \in I^n} \mathbb{k}[z_1, \dots, z_n] e_{\mathbf{i}}$  by the rule:

$$\begin{aligned} y_k \cdot f e_{\mathbf{i}} &= z_k f e_{\mathbf{i}} & e_{\mathbf{i}'} \cdot f e_{\mathbf{i}} &= \delta_{\mathbf{i}, \mathbf{i}'} f e_{\mathbf{i}} \\ \psi_k \cdot f e_{\mathbf{i}} &= \begin{cases} \frac{f^{(k, k+1)} - f}{z_{k+1} - z_k} e_{\mathbf{i}} & i_k = i_{k+1} \\ P_{i_{k+1} i_k}(z_k, z_{k+1}) \cdot f^{(k, k+1)} e_{\mathbf{i}^{(k, k+1)}} & i_k \neq i_{k+1} \end{cases} \end{aligned}$$

and similarly  $R_n$  has a representation of  $Z = \oplus_{\mathbf{j} \in \tilde{I}^n} \mathbb{k}[z_1, \dots, z_n] e_{\mathbf{j}}$  by the same formulas. The same formula as  $\nu$  gives a homomorphism  $\nu_Z: Z \rightarrow \hat{Z}$ :

$$z_k \mathbf{e}_j \mapsto ((u_k^{-1} z_k)^{1/d_{i_k}} - 1) \mathbf{e}_j$$

Note that this map becomes an isomorphism after completion, since  $(u_k^{-1} z_k)^{1/d_{i_k}} - 1 = \frac{1}{d_{i_k}} u_k^{-1} z_k + \dots$ . The homomorphism  $\nu$  is induced by transport of structure via  $\nu_Z$ . The only interesting calculation needed to confirm this is the image of  $\psi_k \mathbf{e}_j$  if  $j_k \neq j_{k+1}$ . From the definition, we have that

$$\psi_k \mathbf{e}_j \cdot f e_{\mathbf{j}} = \prod_{\substack{1/d_{i_{k+1}} = a_{i_{k+1} i_k}^{(m)} \\ u_{k+1}}}^{1/d_{i_k}} \left( (u_{k+1}^{-1} z_{k+1})^{1/d_{i_{k+1}}} - (u_k^{-1} z_k)^{1/d_{i_k}} \right) f^{(k, k+1)} e_{\mathbf{j}^{(k, k+1)}}$$

Equation (3.1) shows that the RHS is the same as  $A_k^{-1} \psi_k \cdot f e_{\mathbf{j}}$ .  $\square$

Let  $\hat{R}$  be the completion of  $R$  with respect to the grading.

**Proposition 3.3** *The map  $\nu$  induces an isomorphism  $\hat{R} \rightarrow \hat{R}$ .*

*Proof.* Let  $A_k$  be the element of  $\hat{R}$  mapping to  $A_k$  under  $\nu$ . That is:

$$A_k \mathbf{e}_j = p_{i_k i_{k+1}} \mathbf{e}_j \prod_{\substack{1/d_{i_k} \neq a_{i_k i_{k+1}}^{(m)} \\ u_k}}^{1/d_{i_k}} (u_k^{1/d_{i_k}} (1 + y_k) - a_{i_k i_{k+1}}^{(m)} u_{k+1}^{1/d_{i_{k+1}}} (1 + y_{k+1})) \prod_{\substack{1/d_{i_k} = a_{i_k i_{k+1}}^{(m)} \\ u_k}}^{1/d_{i_k}} u_k^{1/d_{i_k}}$$

Note that  $A_k$  is invertible in  $\hat{R}$ , since it has non-zero constant term.

The homomorphism  $\nu$  is inverted by the map:

$$\begin{aligned} \mathbf{e}_j &\mapsto \mathbf{e}_j & y_k \mathbf{e}_j &\mapsto u_k (1 + y_k)^{d_{i_k}} \\ (3.2) \quad \psi_k \mathbf{e}_j &\mapsto \begin{cases} A_k \psi_k \mathbf{e}_j & i_k \neq i_{k+1} \\ \frac{1}{u_{k+1} (1 + y_{k+1})^{d_{i_{k+1}}} - u_k (1 + y_k)^{d_{i_k}}} (\psi_k - 1) \mathbf{e}_j & i_k = i_{k+1}, u_k \neq u_{k+1} \\ \psi_k \mathbf{e}_j & j_k = j_{k+1} \end{cases} \end{aligned}$$

Thus the map  $\nu$  is an isomorphism after completion.  $\square$

**3.2. Application to categorical actions.** We've considered the KLR algebra  $R$  with an eye toward studying categorical actions of Lie algebras. By "a categorical action of a Lie algebra" we mean a representation of a specific 2-category  $\mathcal{U}$  defined by Khovanov-Lauda [KL10] and Rouquier [Roua]. We'll follow the conventions of [Weba, 2.4], where this 2-category is presented by taking the quotient of a 2-category  $\tilde{\mathcal{U}}$  of KL diagrams (see [Weba, 2.3]) by the KLR relations (2.1a–2.1g) and the following relations on 2-morphisms:

$$(3.3a) \quad \begin{array}{c} i \quad j \quad i \\ \swarrow \quad \searrow \\ j \end{array} = \begin{array}{c} i \quad j \quad i \\ \swarrow \quad \searrow \\ j \end{array} \quad \begin{array}{c} j \\ \swarrow \quad \searrow \\ i \quad j \quad i \end{array} = \begin{array}{c} j \\ \swarrow \quad \searrow \\ i \quad j \quad i \end{array}$$

$$(3.3b) \quad \begin{array}{c} i \quad j \quad i \\ \swarrow \quad \searrow \\ j \end{array} = t_{ij} \begin{array}{c} i \quad j \quad i \\ \swarrow \quad \searrow \\ j \end{array} \quad \begin{array}{c} j \\ \swarrow \quad \searrow \\ i \quad j \quad i \end{array} = t_{ij} \begin{array}{c} j \\ \swarrow \quad \searrow \\ i \quad j \quad i \end{array}.$$

$$(3.3c) \quad \sum_{k=\lambda^i-1}^{j+\lambda^i+1} \begin{array}{c} \text{circle with dot at } k \\ \lambda \end{array} j-k = \begin{cases} 1 & j = -2 \\ 0 & j > -2 \end{cases}$$

$$(3.3d) \quad \lambda \begin{array}{c} \text{figure-eight} \end{array} = - \sum_{a+b=-1} \begin{array}{c} \text{figure-eight with dots at } a, b \end{array} \quad \lambda \begin{array}{c} \text{figure-eight} \end{array} = \sum_{a+b=-1} \begin{array}{c} \text{figure-eight with dots at } a, b \end{array}$$

$$(3.3e) \quad \lambda \begin{array}{c} \text{figure-eight} \end{array} = \lambda \begin{array}{c} \text{two parallel lines} \end{array} + \sum_{a+b+c=-2} \begin{array}{c} \text{figure-eight with dots at } a, b, c \end{array} \lambda$$

$$(3.3f) \quad \lambda \begin{array}{c} \text{figure-eight} \end{array} = \lambda \begin{array}{c} \text{two parallel lines} \end{array} + \sum_{a+b+c=-2} \begin{array}{c} \text{figure-eight with dots at } a, b, c \end{array} \lambda$$



*Proof.* (1)  $\Rightarrow$  (2): We have already discussed the existence of an appropriate  $R_n$  action, based on Proposition 3.3.

By [Roub, 4.18], it suffices to check the  $\mathfrak{sl}_2$  relations hold. That is, the proof for general  $\mathfrak{g}$  will follow from the result for  $\mathfrak{g} = \mathfrak{sl}_2$  and  $I = \{i\}$  a single dot. In this case,  $\tilde{I}$  is disconnected and  $\tilde{\mathfrak{g}} \cong \mathfrak{sl}_2^{\oplus \ell}$  where  $\ell = \#U_i$ . We pick a bijection  $\tilde{I} \cong [1, \ell]$

Furthermore, by [Roua, 5.5], it suffices to consider the universal categorification of the simple  $\mathfrak{sl}_2^{\oplus \ell}$ -module with highest weight  $(n_1, \dots, n_\ell)$ ; in fact, since  $y$  is nilpotent, we need only consider the universal categorification where  $y$  is topologically nilpotent. Consider the completion of the deformed cyclotomic quotient  $\check{R}^{(n_1, \dots, n_\ell)}$ , the quotient of  $R \otimes_{\mathbb{k}} \mathbb{k}[[q_{1,1}, \dots, q_{1,n_1}, \dots, q_{\ell,1}, \dots, q_{\ell,n_\ell}]]$  by the relations

$$p_{m_1}(y_1 e_{\mathbf{m}}) = 0 \quad \text{with} \quad p_m(u) := u^{n_{m_1}} + q_{m_1,1} u^{n_{m_1}-1} + \dots + q_{m_1,n_{m_1}}.$$

This is Morita equivalent to the tensor product of analogous completions for the points of  $\tilde{I}$ :  $\check{R}^{n_1} \otimes \dots \otimes \check{R}^{n_\ell}$ . The projective modules over  $\check{R}^{(n_1, \dots, n_\ell)}$  are universal amongst categorification where  $y$  is nilpotent.

Consider the quotient  $R[[\mathbf{q}]]/p(y_1)$  with  $p(w) = \prod_{m=1}^{\ell} p_m(u_m^{-1}w - 1)$ . The map  $v^{-1}$  induces a map of  $\mathbb{k}[[\mathbf{q}]]$ -algebras  $R[[\mathbf{q}]] \rightarrow \check{R}^{(n_1, \dots, n_\ell)}$ . This map sends  $p(y_1)e_{\mathbf{m}}$  to

$$p(u_m(1 + y_1))e_{\mathbf{m}} = p_1(u_1^{-1}u_m y_1 + u_1^{-1}u_m - 1) \dots p_m(y_1) \dots p_\ell(u_\ell^{-1}u_m y_1 + u_\ell^{-1}u_m - 1)e_{\mathbf{m}} = 0.$$

Thus, this induces a map  $R[[\mathbf{q}]]/p(y_1) \rightarrow \check{R}^{(n_1, \dots, n_\ell)}$ .

On the other hand,  $v$  induces a map  $\hat{R}[[\mathbf{q}]] \rightarrow R[[\mathbf{q}]]/p(y_1)$  (since the spectrum of  $y$  in the latter quotient lies in  $U$ ). This map sends

$$p_{m_1}(y_1 e_{\mathbf{m}}) \mapsto p_{m_1}((u_{m_1}^{-1}y_1 - 1)e_{\mathbf{m}}) = p(y_1) \cdot \prod_{m \neq m_1} p_m(u_m^{-1}y_1 - 1)^{-1}e_{\mathbf{m}} = 0.$$

Thus, we obtain an isomorphism  $\check{R}^{(n_1, \dots, n_\ell)} \cong \hat{R}[[\mathbf{q}]]/p(y_1)$ . Thus, the action on  $\check{R}^{(n_1, \dots, n_\ell)}$  is induced by the action on  $R[[\mathbf{q}]]/p(y_1)$ -modules described in [Weba, 3.25].

(2)  $\Rightarrow$  (1): By assumption, the spectrum of  $y$  on  $\mathcal{E}_i$  is contained in  $U_i$ . Thus, the action of  $R_n$  on  $\mathcal{E}^n$  extends to an action of  $\hat{R}_n$ , and thus of  $\check{R}_n$  by Proposition 3.3. As in the proof the other direction, it's enough to check that each  $\mathcal{E}_{i,u}, \mathcal{F}_{i,u}$  give a categorical  $\mathfrak{sl}_2$ -action, and thus enough to consider  $\mathfrak{g} = \mathfrak{sl}_2$ . Consider an indecomposable object  $B$  which is highest weight for  $\mathcal{E}_i$ ; the category generated by  $B$  under the action of  $\mathfrak{sl}_2$  is equivalent to  $\check{R}^n \otimes_{\check{R}_n} \text{End}(B)$  by [Weba, 3.25]. By replacing this subcategory with its module category, we can assume that  $B$  is simple. In this case, the subcategory additively generated by categorification functors acting on  $B$  is equivalent to  $\check{R}/p(y_1)$  for some polynomial  $p(u) \in K[u]$  with coefficients given by the action of the fake clockwise bubbles. By (3.5), this polynomial must be  $p(w) = \prod_{m=1}^{\ell} (w - u_m)^{n_m}$  where  $n_m = \alpha_{(i,m)}^\vee(\lambda)$ . As argued before,  $\check{R}/p(y_1) \cong \check{R}^{(n_1, \dots, n_m)}$ , so we obtain the desired  $\mathfrak{sl}_2^{\oplus \ell}$ -action.

Now, for a fixed object  $B$  (no longer assumed to be highest weight), we wish to establish the result for the subcategory generated by  $B$  under the  $\mathfrak{g}$  categorification

functors. We establish this by induction on the smallest integer  $p$  such that  $\mathcal{E}_i^p B = 0$ . If this is the case for  $p = 1$ , then we are back in the highest weight object case. Otherwise, induction shows that the object  $\mathcal{E}_i B$  generates a subcategory  $C'$  where the map  $\rho_{(i,m),\lambda}$  induces an isomorphism. Thus, it suffices to check this natural transformation is an isomorphism in the quotient category  $C/C'$ . In this quotient,  $B$  is highest weight, so we are done. This establishes that we have the desired  $\mathfrak{sl}_2^{\oplus \ell}$ -action, which as argued above gives the  $\tilde{\mathfrak{g}}$  action.  $\square$

#### 4. DEFORMED TENSOR PRODUCT ALGEBRAS

**4.1. The definition.** In [Web15, §5], we introduced a natural categorifications  $\mathcal{X}^\lambda$  for tensor products of highest and lowest weight representations. These categorifications have natural deformations, which we wish to study in the context of the previous section.

We will use the notation



to denote the endomorphism  $(y + b)^a$  of  $\mathcal{F}_i$ , and similarly for  $\mathcal{E}_i$ .

In [Web15, §4], we introduced a notion of **tricolore diagrams**, which naturally form the 2-morphisms of a 2-category  $\mathcal{T}$ . The categorifications  $\mathcal{X}^\lambda$  are natural subquotients of this category, and our deformed categorifications arise from a straightforward deformation of the relations from [Web15], which we present below:

**Definition 4.1** *Let  $\mathcal{T}$  be the quotient of  $\tilde{T} \otimes \mathbb{k}[z_1, \dots, z_\ell]$  by the relations (2.1a–2.1g, 3.3a–3.3g) on black strands and (4.1a–4.1g) below relating red and blue strands to black. Note that the relations (4.1a–4.1g) are deformations of the relations of  $\mathcal{T}$  in [Web15, 4.3]; we will thus recover the category  $\mathcal{T}$  if we specialize  $z_i = 0$ .*

$$(4.1a) \quad \lambda \begin{array}{c} \text{blue crossing} \\ \text{strand } i \text{ (bottom-left), strand } -\mu \text{ (bottom-right)} \end{array} = \lambda \begin{array}{c} \text{blue strands} \\ \text{strand } i \text{ (left), strand } -\mu \text{ (right)} \end{array} \quad \lambda \begin{array}{c} \text{red crossing} \\ \text{strand } i \text{ (bottom-left), strand } \mu \text{ (bottom-right)} \end{array} = \lambda \begin{array}{c} \text{red strands} \\ \text{strand } i \text{ (left), strand } \mu \text{ (right)} \end{array}$$



(4.1b)

$\lambda_k$   $i$   $=$   $\lambda_k$   $i$   $-z_k$   $\lambda_k^i$   $=$   $-\lambda_k$   $i$   $-z_k$   $\lambda_k^i$   
 $i$   $\lambda_k$   $i$   $\lambda_k^i$   $-z_k$   $i$   $-z_k$   $\lambda_k^i$

(4.1c)

$\lambda_k$   $\lambda_k$   $=$   $\lambda_k$   $\lambda_k$   $-\lambda_k$   $-\lambda_k$   $=$   $-\lambda_k$   $-\lambda_k$   $-\lambda_k$   $-\lambda_k$

(4.1d)

$\lambda_k$   $\lambda_k$   $=$   $\lambda_k$   $\lambda_k$   $-\lambda_k$   $-\lambda_k$   $=$   $-\lambda_k$   $-\lambda_k$   $-\lambda_k$   $-\lambda_k$

(4.1e)

$\lambda_k$   $\lambda_k$   $=$   $\lambda_k$   $\lambda_k$   $-\lambda_k$   $-\lambda_k$   $=$   $-\lambda_k$   $-\lambda_k$   $-\lambda_k$   $-\lambda_k$

$$\begin{aligned}
 (4.1f) \quad & \begin{array}{c} \text{Diagram: A crossing of two black strands. The left strand is labeled } j \text{ and the right strand is labeled } i. \text{ A blue strand enters from the top left, loops around the crossing, and exits at the bottom right. The label } -\lambda_k \text{ is placed below the crossing.} \end{array} = \begin{array}{c} \text{Diagram: A crossing of two black strands. The left strand is labeled } j \text{ and the right strand is labeled } i. \text{ A blue strand enters from the top right, loops around the crossing, and exits at the bottom left. The label } -\lambda_k \text{ is placed below the crossing.} \end{array} - \sum_{a+b-1=\lambda_k^i} \delta_{i,j} \begin{array}{c} \text{Diagram: A blue strand enters from the top, passes through a circle labeled } -z_k, \text{ and exits at the bottom. The label } j \text{ is below the entry point.} \end{array} \begin{array}{c} \text{Diagram: A vertical blue strand. The label } -\lambda_k \text{ is placed to its left.} \end{array} \begin{array}{c} \text{Diagram: A blue strand enters from the top, passes through a circle labeled } -z_k, \text{ and exits at the bottom. The label } i \text{ is below the entry point.} \end{array} a \\
 (4.1g) \quad & \begin{array}{c} \text{Diagram: A crossing of two black strands. The left strand is labeled } j \text{ and the right strand is labeled } i. \text{ A red strand enters from the top left, loops around the crossing, and exits at the bottom right. The label } \lambda_k \text{ is placed below the crossing.} \end{array} = \begin{array}{c} \text{Diagram: A crossing of two black strands. The left strand is labeled } j \text{ and the right strand is labeled } i. \text{ A red strand enters from the top right, loops around the crossing, and exits at the bottom left. The label } \lambda_k \text{ is placed below the crossing.} \end{array} + \sum_{a+b-1=\lambda_k^i} \delta_{i,j} \begin{array}{c} \text{Diagram: A red strand enters from the top, passes through a circle labeled } -z_k, \text{ and exits at the bottom. The label } j \text{ is below the entry point.} \end{array} \begin{array}{c} \text{Diagram: A vertical red strand. The label } \lambda_k \text{ is placed to its left.} \end{array} \begin{array}{c} \text{Diagram: A red strand enters from the top, passes through a circle labeled } -z_k, \text{ and exits at the bottom. The label } i \text{ is below the entry point.} \end{array} a
 \end{aligned}$$

The reader should read the label  $\lambda_k$  in this diagram to indicate that the strand shown is the  $k$ th of the red and blue strands from the left. In particular,  $z_k$  is connected to this  $k$ th strand, and could be thought of as a new endomorphism of the tricolore triple with a single red or blue strand and  $\mathbf{i} = \emptyset$ .

We let  $\mathcal{X}^\Delta$  be the idempotent completion of the quotient of the category of tricolore quadruples  $(\underline{\lambda}, \mathbf{i}, \kappa, \mathcal{L})$  in  $\mathcal{T}$  by the tricolore quadruples where  $\kappa(1) > 0$ . That is, we consider 1-morphisms with label 0 at the left, where we fix the labels of the red and blue strands as well as their order to match  $\underline{\lambda}$ , but allow arbitrary black strands. We then take the quotient of this category of 1-morphisms by killing the diagrams with a black line at the far left.

The definition of  $\mathcal{X}^\Delta$  has precisely the same form as that of  $\mathcal{X}^\Delta$ , with the only difference being the relations (4.1a–4.1g) in place of the relations in [Web15, 4.3].

From the definition, it's clear that there is a 2-functor  $\mathcal{U} \rightarrow \mathcal{T}$ , since (3.3a–3.3g) are simply the relations of  $\mathcal{U}$ . Thus, composition on the right induces a  $\mathcal{U}$  action on  $\mathcal{X}^\Delta$ .

**Definition 4.2** Given a  $\mathbb{k}[z_1, \dots, z_\ell]$ -algebra  $K$ , we let  $\mathcal{X}_K^\Delta$  be the idempotent completion of the extension of scalars  $\mathcal{X}^\Delta \otimes_{\mathbb{k}[z_1, \dots, z_\ell]} K$ .

The main examples we'll want to consider are  $\mathbb{K} = \mathbb{k}(z_1, \dots, z_m)$  and the algebraic closure  $\bar{\mathbb{K}}$ .

## 4.2. Spectral analysis.

**Definition 4.3** Define sets  $U_i \subset \bar{\mathbb{K}}$  as follows: if  $\alpha_i^\vee(\lambda_m) \neq 0$ , then  $z_m \in U_i^{(0)}$ , and all elements of  $U_i^{(0)}$  are of this type. Now we inductively define  $U_i^{(N)}$  to be the union of  $U_i^{(N-1)}$  with the elements  $u$  of  $\mathbb{k}$  that satisfy  $Q_{ij}(u, u') = 0$  for  $u' \in U_j^{(N-1)}$ .

Let  $U'_i$  be the union of  $U_i$  with the set of elements in  $\bar{\mathbb{K}}$  that appear in the spectrum of the elements  $y_k e_i$  with  $i_k = i$  the category  $\mathcal{X}_{\bar{\mathbb{K}}}^\Delta$ ; that, is the eigenvalues that appear when dots on strands with label  $i$  act.

It might seem strange that we add the elements of  $U_i$  to  $U'_i$  by definition, but this simplifies matters for us, since we have not yet established that  $\mathcal{X}_{\mathbb{K}}^{\Lambda}$  is non-zero. Thus, we have not yet established that there are any elements of this spectrum. We will ultimately see that  $U_i = U'_i$ , and these both coincide with the union of spectra discussed above.

We let  $\tilde{I}', \tilde{I}$  be the graphs constructed from these sets as before and  $\tilde{\mathfrak{g}}'$  and  $\tilde{\mathfrak{g}}$  the corresponding Kac-Moody algebras. We let  $\tilde{\lambda}$  be a weight for  $\tilde{\mathfrak{g}}'$  such that  $\alpha_{(i,u)}^{\vee}(\tilde{\lambda}) = \delta_{u,z_m} \alpha_i^{\vee}(\lambda_m)$ .

Since the elements  $z_m$  are algebraically independent from each other, every element of  $u \in U_i$  is algebraically dependent on exactly one  $z_m$ . We denote this index  $m(u)$ .

We can define formal power series valued in the center of the category  $\mathcal{X}^{\Lambda}$  which act on the object  $(\underline{\lambda}, \mathbf{i}, \kappa)$  as

$$\mathbf{y}_i(w) := \prod_{i_k = \pm i} (1 + w y_k)^{\pm 1} \quad \mathbf{Q}_{ji}(w) = \prod_{i_k = \pm j} \left( t_{ij}^{-1} \cdot (-w)^{-c_{ij}} Q_{ji}(y_k, -w^{-1}) \right)^{\pm 1},$$

where  $y_k$  is the dot acting on the  $k$ th strand from the left.

These are supersymmetric polynomials (in the sense of [Ste85]) in the two alphabets of dots on upward oriented  $i$ -strands and dots on downward oriented  $i$ -strands. Any such polynomial commutes with all upward or downward oriented diagrams by [KL09, 2.9], since each coefficient is symmetric in the corresponding variables. It commutes with a cup or cap joining the  $k$ th strand to the  $k + 1$ st since multiplying by  $(1 + u y_k)^{\pm 1}$  at one end of the cup or cap cancels with  $(1 + u y_{k+1})^{\mp 1}$  at the other (this is a restatement of the supersymmetric property). Note that the bubble slides and triviality of bubbles at the far left show that

$$(4.2) \quad \sum_{k=0}^{\infty} i \text{ (strand with } k \text{ dots)} w^k = \mathbf{y}_i(w)^2 \prod_{j \neq i} \mathbf{Q}_{ji}(w)^{-1} \prod_{m=1}^{\ell} (w - z_m)^{\lambda_m^i}.$$

Let  $\tilde{\mu} = \tilde{\lambda} - \sum a_{i,u} \alpha_{i,u}$  be a weight of  $\tilde{I}'$ .

**Definition 4.4** We let  $\mathcal{V}_{\tilde{\mu}}$  be the subcategory of  $\mathcal{X}_{\mathbb{K}}^{\Lambda}$  on which every coefficient of  $\mathbf{y}_i(w) - \prod_{u \in U'_i} (1 + w u)^{-a_{i,u}}$  acts nilpotently.

Note that by the definition of  $U'_i$ , we must have that  $\mathcal{X}_{\mathbb{K}}^{\Lambda} \cong \oplus_{\tilde{\mu}} \mathcal{V}_{\tilde{\mu}}$ .

**Lemma 4.5** The functors  $\mathcal{F}_{i,u}$  and their adjoints  $\mathcal{E}_{i,u}$  induce a categorical action of  $\tilde{\mathfrak{g}}'$  on  $\mathcal{X}_{\mathbb{K}}^{\Lambda}$ , with weight decomposition given by  $\mathcal{X}_{\mathbb{K}}^{\Lambda} \cong \oplus_{\tilde{\mu}} \mathcal{V}_{\tilde{\mu}}$ .

*Proof.* We'll use Theorem 3.4 (with the role of  $\mathbb{k}$  taken by  $\mathbb{K}$ ). By assumption, any eigenvalue of  $y$  acting on  $\mathcal{E}_i$  must appear as a pole or zero of  $\prod_{u \in U'_i} (1 + w u)^{-a_{i,u}}$ , and thus lie in  $U'_i$ . Next, we should note that by (4.2), we can compute the semi-simple

part of the action of the bubbles on an object in  $\mathcal{V}_{\tilde{\mu}}$ . Consider the order of the pole or root that this power series has at  $u$ . By assumption the term of  $y_i^2$  contributes  $-2a_{i,u}$ ; the term  $\mathbf{Q}_{ji}(u)^{-1}$  contributes  $a_{j,u'}$  for each edge joining  $(i, u)$  to  $(j, u')$  in  $\tilde{I}$ . Thus, we obtain that this order is

$$\sum_{m=1}^{\ell} \delta_{z_m, u} \alpha_i^{\vee}(\lambda_m) - \sum_{j, u'} \alpha_{i, u}^{\vee}(\alpha_{j, u'}) = \alpha_{i, u}^{\vee}(\tilde{\mu}).$$

This matches with (3.5), and thus confirms the hypotheses of condition (2) of Theorem 3.4. The result follows.  $\square$

Given a triple  $(\underline{\lambda}, \mathbf{i}, \kappa)$  with  $\mathbf{i} = (i_1, \dots, i_n)$  considered as an object in  $\mathcal{X}_{\mathbb{K}}^{\underline{\lambda}}$  (recall that we will often exclude  $\underline{\lambda}$  from the notation when it is unlikely to be confused), we can thus decompose it according to the spectrum of the dots  $y_k$ . For a sequence  $j_k = (i_k, u_k) \in \tilde{I}'$  for  $k = 1, \dots, n$ , we let  $(\mathbf{i}, \kappa)_{\mathbf{u}}$  be the simultaneous stable kernel of  $y_k - u_k$  for all  $1 \leq k \leq n$ .

**Lemma 4.6** *We have that*

$$(4.3) \quad (\mathbf{i}, \kappa)_{\mathbf{u}} \cong \mathcal{E}_{i_n, u_n} \cdots \mathcal{E}_{i_1, u_1}(\emptyset, 0)$$

if  $u_k \in U_{i_k}$  and  $k > \kappa(m(u_k))$  for each  $k$ , and  $(\mathbf{i}, \kappa)_{\mathbf{u}} = 0$  otherwise.

In particular, we have  $U_i = U'_i$  for all  $i$  and  $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}'$ , and the category  $\mathcal{X}_{\mathbb{K}}^{\underline{\lambda}}$  is generated by the tricolore triple  $(\emptyset, 0)$  as a categorical module over  $\tilde{\mathfrak{g}}$ .

*Proof.* First, we note that if  $u \in U_i$  and  $u' \notin U_j$ , then  $Q_{ij}(y_1, y_2)$  acts on  $\mathcal{E}_{i, u} \mathcal{E}_{j, u'}$  with its only eigenvalue  $Q_{ij}(u, u') \neq 0$  (by the definition of  $U_i$ ). Thus the crossing  $\psi$  induces an isomorphism  $\mathcal{E}_{i, u} \mathcal{E}_{j, u'} \cong \mathcal{E}_{j, u'} \mathcal{E}_{i, u}$ . Similarly,  $\mathcal{F}_{j, u'}$  commutes past all red and blue strands since  $y - z_k$  is invertible, with its only eigenvalue  $u' - z_k$ ; in fact, this still follows for the  $k$ th red/blue strand if  $u' \neq z_k$  (in particular, if  $k \neq m(u')$ ).

We establish the result by induction on  $\ell$  and  $n$  (i.e. on the total number of strands). If  $n = 0$ , the result is tautological. Otherwise, the rightmost strand in the idempotent for the object  $(\mathbf{i}, \kappa)$  is either black, blue or red. If it is black, then  $(\mathbf{i}, \kappa) = \mathcal{E}_i(\mathbf{i}^-, \kappa)$  for some  $i \in \pm I$ , and decomposing with respect to the eigenvalues of  $y$ , we have  $\mathcal{E}_i(\mathbf{i}^-, \kappa) \cong \oplus \mathcal{E}_{i, u}(\mathbf{i}^-, \kappa)$  where  $u$  ranges over the roots of the minimal polynomial of  $y$  acting on  $\mathcal{E}_i(\mathbf{i}^-, \kappa)$ . By induction  $(\mathbf{i}^-, \kappa)$  is a sum of modules obtained from  $(\emptyset, 0)$  by the functors  $\mathcal{E}_{j, u'}$  and  $\mathcal{F}_{j, u'}$  for  $u' \in U_j$ . If  $u$  is not in  $U_i$ , then all these functors commute with  $\mathcal{E}_{i, u}$  (as argued above), and  $\mathcal{E}_{i, u}(\emptyset, 0) = 0$ , so  $\mathcal{E}_{i, u}(\mathbf{i}^-, \kappa) = 0$ , and  $\mathcal{F}_i(\mathbf{i}^-, \kappa) \cong \oplus_{u \in U_i} \mathcal{E}_{i, u}(\mathbf{i}^-, \kappa)$ . By induction, this establishes the result.

On the other hand, if the rightmost strand is blue or red, we simply apply induction with the tricolore triple  $(\underline{\lambda}^-, \mathbf{i}, \kappa^-)$  with this strand removed. By induction,  $(\underline{\lambda}^-, \mathbf{i}, \kappa^-) \cong \oplus (\underline{\lambda}^-, \mathbf{i}, \kappa^-)_{\mathbf{u}}$  with  $k > \kappa(m(u_k))$  and  $m(u_k) < \ell$ . Since adding in the  $\ell$ th blue or red strand does not change the eigenvalues of the dots, we also have  $(\underline{\lambda}, \mathbf{i}, \kappa) \cong \oplus (\underline{\lambda}, \mathbf{i}, \kappa)_{\mathbf{u}}$  with  $\mathbf{u}$  ranging over the same set. This shows equation (4.3), and that  $U'_i = U_i$ .  $\square$

Thus  $\mathcal{X}_{\mathbb{K}}^{\lambda}$  is generated by a single object, which is highest weight for the components of  $\tilde{I}$  with  $\lambda_{m(u)}$  dominant and lowest weight for those with  $\lambda_{m(u)}$  anti-dominant. Alternatively, we can easily choose a Borel for which this representation is straightforwardly highest weight. To distinguish objects which are highest weight for this Borel, we call them **signed highest weight**. Since  $\tilde{I}$  is split into components in bijection with  $[1, m]$ , we can write each weight  $\tilde{\lambda}$  uniquely as a sum  $\tilde{\lambda} = \tilde{\lambda}_1 + \dots + \tilde{\lambda}_m$  where  $\tilde{\lambda}_m$  is supported on the component with  $m(u) = m$ .

We can apply the classification of highest weight representations in [Weba, 3.25], which shows that any idempotent complete representation additively generated by a highest weight object is equivalent to the projective modules over a base change of the deformed cyclotomic quotient  $\check{R}^{\tilde{\lambda}}$  for  $\check{\mathfrak{g}}$  of highest weight  $\tilde{\lambda}$ . This base change is by the endomorphism ring of the highest weight object over the ring  $\check{R}_{\tilde{\lambda}}^{\tilde{\lambda}}$ , the polynomial ring on fake bubbles in weight  $\tilde{\lambda}$ .

We can justify the change of Borel by applying the Cartan involution in the factors where  $\lambda_{m(u)}$  is anti-dominant, which is essentially relabeling  $\mathcal{F}_{i,u}$  as  $\mathcal{E}_{i,u}$  instead. The ring  $\check{R}^{\tilde{\lambda}}$  is Morita equivalent to the tensor product  $\check{R}^{\tilde{\lambda}_1} \otimes \dots \otimes \check{R}^{\tilde{\lambda}_\ell}$ . The different choice of Borel means that it is more convenient to think of deformed cyclotomic quotient  $\check{R}^{\tilde{\lambda}_k}$  as written with a red strand and downward black strands if  $\lambda_k$  is dominant, and with a blue strand and upward black strands if  $\lambda_k$  is anti-dominant.

Thus, we find that  $\mathcal{X}_{\mathbb{K}}^{\lambda}$  is the base change of  $\check{R}^{\tilde{\lambda}}$ -pmod via the natural map  $\check{R}_{\tilde{\lambda}}^{\tilde{\lambda}} \rightarrow \text{End}(\emptyset, 0)$  to the endomorphisms of the signed highest weight object. Since  $\mathbf{y}: \mathcal{F}_{i,u} \rightarrow \mathcal{F}_{i,u}$  is nilpotent and the endomorphisms of  $\text{End}(\emptyset, 0)$  are just the scalars, this resulting base change is simply the tensor product of undeformed cyclotomic quotients  $\mathcal{X}^{\tilde{\lambda}} \otimes_{\mathbb{K}} \mathbb{K} \cong \mathcal{X}^{\tilde{\lambda}_1} \otimes \mathcal{X}^{\tilde{\lambda}_2} \otimes \dots \otimes \mathcal{X}^{\tilde{\lambda}_\ell} \otimes_{\mathbb{K}} \mathbb{K}$ . Thus, we have an induced strongly equivariant functor

$$(4.4) \quad \Phi: \mathcal{X}^{\tilde{\lambda}} \otimes_{\mathbb{K}} \mathbb{K} \rightarrow \mathcal{X}_{\mathbb{K}}^{\lambda}.$$

Since this functor is strongly equivariant, and both source and target are generated by a signed highest weight object with only scalar endomorphisms, this functor is either an equivalence or  $\mathcal{X}_{\mathbb{K}}^{\lambda}$  is a trivial category. In particular,  $\mathcal{X}_{\mathbb{K}}^{\lambda}$  satisfies any hypotheses that  $\mathcal{X}^{\tilde{\lambda}} \otimes_{\mathbb{K}} \mathbb{K}$  does. For example, each weight space is equivalent to the category of projective modules over a finite dimensional  $\mathbb{K}$ -algebra.

It will actually be more convenient for us to think of the irreducible module with signed highest weight  $\tilde{\lambda}$  as a tensor product of simples corresponding to the different components. That is, we use the equivalence of  $\mathcal{X}^{\tilde{\lambda}}$  with the tensor product category  $\mathcal{X}^{\underline{\tilde{\lambda}}}$  where  $\underline{\tilde{\lambda}} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_\ell)$ . We can define a functor

$$\Xi: \mathcal{X}_{\mathbb{K}}^{\lambda} \rightarrow \mathcal{X}^{\underline{\tilde{\lambda}}} \otimes_{\mathbb{K}} \mathbb{K}$$

which sends  $(\mathbf{i}, \kappa)$  to the sum  $\bigoplus (\mathbf{j}, \kappa)$  where  $\mathbf{j}$  ranges over sequences with  $j_k = (i_k, u_k)$ . If  $\mathbf{j}$  satisfies  $k > \kappa(m(u_k))$ , then we have  $(\mathbf{j}, \kappa) \cong (\mathbf{j}, 0)$ , and otherwise the corresponding

object is 0, so Lemma 4.6 shows that this is quasi-inverse to  $\Phi$  on the level of 1-morphisms.

Now, we must define how  $\Xi$  acts on 2-morphisms. First, note that by Theorem 3.4, we have a categorical action of  $\mathfrak{g}$  on  $\mathcal{X}^{\tilde{\Lambda}} \otimes_{\mathbb{K}} \mathbb{K}$ . On purely black diagrams,  $\Xi$  simply employs this action; that is, on upward oriented diagrams, it follows the formula (3.2). Since left (or right) adjunctions are unique up to isomorphism, we can send the leftward cup and cap in  $\mathcal{U}_{\mathfrak{g}}$  to any adjunction we choose. For simplicity, we simply match leftward oriented cups and caps as below:

$$(4.5) \quad \begin{array}{c} i \\ \curvearrowright \\ i \end{array} \mapsto \sum_{u \in U_i} \begin{array}{c} (i, u) \\ \curvearrowright \\ (i, u) \end{array} \quad \begin{array}{c} i \\ \curvearrowleft \\ i \end{array} \mapsto \sum_{u \in U_i} \begin{array}{c} (i, u) \\ \curvearrowleft \\ (i, u) \end{array}$$

For rightward oriented cups, the formula is quite complicated, but is fixed by the choices we have made thus far, and the existence of a consistent choice follows from the existence of the  $\mathfrak{g}$ -action. Thus, we need only show this action on diagrams with red/blue strands, with formulas given below (4.6–4.7).

$$(4.6) \quad \begin{array}{c} \text{red} \\ \diagdown \\ i \end{array} \lambda_m \mapsto \sum_{u \in U_i} \begin{array}{c} \text{red} \\ \diagdown \\ (i, u) \end{array} \lambda_m \quad \begin{array}{c} \text{red} \\ \diagup \\ \lambda_m \end{array} i \mapsto \begin{array}{c} \text{red} \\ \diagup \\ \lambda_m \end{array} (i, z_m) + \sum_{u \in U_i \setminus \{z_m\}} (y - z_m + u)^{\lambda_m^i} \begin{array}{c} \text{red} \\ \diagup \\ \lambda_m \end{array} (i, u)$$

$$(4.7) \quad \begin{array}{c} \text{blue} \\ \diagup \\ i \end{array} \lambda_m \mapsto \sum_{u \in U_i} \begin{array}{c} \text{blue} \\ \diagup \\ (i, p) \end{array} \lambda_m \quad \begin{array}{c} \text{blue} \\ \diagdown \\ \lambda_m \end{array} i \mapsto \begin{array}{c} \text{blue} \\ \diagdown \\ \lambda_m \end{array} (i, z_m) + \sum_{u \in U_i \setminus \{z_m\}} (y - z_m + u)^{\lambda_m^i} \begin{array}{c} \text{blue} \\ \diagdown \\ \lambda_m \end{array} (i, u)$$

**Lemma 4.7** *The functor  $\Xi$  is well-defined.*

*Proof.* In order to check this, we have to verify all the relations on 2-morphisms. The equations that can be stated purely using upward or down diagrams, that is, (2.1a–2.1g, 4.1b, 4.1f–4.1d) all follow by straightforward calculations as in the proof of Proposition 3.3.

Thus we only need to argue for the relations involving right cups and caps. The way we have defined the right cup/cap means that relations (3.3a–3.3g) are automatic. The remaining relations (4.1e, 4.1a) are actually redundant when the right cap and cup are defined in terms of the left cup and cap. The relation (4.1e) is the definition of the upward red/black or downward blue/black crossings, in this perspective. For the relation (4.1a), assume we are considering the red version; the blue one follows similarly. We must consider two different cases. Let  $\mu$  be the label of the region at the left of the picture.

- If  $\mu^i \geq 0$ , then we have we have make a loop at the left with  $\mu^i$  dots. Pulling this through and applying (4.1b), then undoing this bubble, we obtain the desired relation.

- if  $\mu^i \leq 0$ , then we start with the diagram with a leftward cup at the bottom and rightward cup at top, and compare the result of applying [Web15, 3.6d] to these two strands to the left and right of the red strand. Using the relations (4.1d, 4.1g), we can move the bigon to the right side of the red line, and using (4.1b) to remove bigons between red and downward strands. The left- and right-hand sides now have the same pattern of black strands, but in one the upward strands make a bigon with the red strand and in the other, they don't. This can only hold if (4.1a) is true.  $\square$

**Lemma 4.8** *The category  $\mathcal{X}_{\mathbb{K}}^{\lambda}$  is equivalent to  $\mathcal{X}^{\bar{\lambda}} \otimes_{\mathbb{K}} \bar{\mathbb{K}}$  via the functor  $\Phi$  defined in (4.4).*

*Proof.* By [Weba, 3.25], this functor is an equivalence if  $\text{End}(\emptyset, 0) \cong \mathbb{K}$ . Thus, the only issue is that the object  $(\emptyset, 0)$  might simply be 0 (in which case the entire category  $\mathcal{X}_{\mathbb{K}}^{\lambda}$  is 0).

The functor  $\Xi$  sends  $(\emptyset, 0)$  to  $(\emptyset, 0)$ . The existence of this functor establishes that  $\mathcal{X}_{\mathbb{K}}^{\lambda}$  is not 0, so  $\Phi$  must be an equivalence. In fact, we can easily see that  $\Xi$  is strongly equivariant for  $\tilde{\mathfrak{g}}$ , so it must be quasi-inverse to  $\Xi$  when composed with the equivalence  $\mathcal{X}^{\bar{\lambda}} \otimes_{\mathbb{K}} \bar{\mathbb{K}} \cong \mathcal{X}^{\bar{\lambda}} \otimes_{\mathbb{K}} \bar{\mathbb{K}}$ .  $\square$

**4.3. Applications.** Now, we turn to the application of our results, culminating in the proof of Theorem A.

**Theorem 4.9** *For two tricolore triples  $(\underline{\lambda}, \mathbf{i}, \kappa)$  and  $(\underline{\lambda}, \mathbf{i}', \kappa')$ , the set  $D$  is a basis over  $\mathbb{C}[\mathbf{z}]$  for the morphism space  $\text{HOM}_{\mathcal{T}}((\underline{\lambda}, \mathbf{i}, \kappa), (\underline{\lambda}, \mathbf{i}', \kappa'))$ .*

*Proof.* The proof that these are a spanning set is essentially equivalent to that of [KL10, Prop. 3.11]. First, note that any two minimal diagrams for the same matching are equivalent modulo those with fewer crossings (using the relations (2.1f, 2.1g, 4.1f, 4.1g)). Similarly, moving dots to the chosen positions only introduces diagrams with fewer crossings.

Thus, we only need show that all minimal diagrams span. Of course, if a diagram is non-minimal then it can be rewritten in terms of the relations in terms of ones with fewer crossings, using the relations to clear all strands out from a bigon, and then the relations (2.1e, 3.3e–3.3f, 3.3g, 4.1b) to remove it. Thus, by induction, this process must terminate at a expression in terms of minimal diagrams. Thus, these elements span, and it suffices to show that these elements are linearly independent when  $\mathbf{z}$  are generic, that is, after base change to  $\bar{\mathbb{K}}$ .

Assume  $\mathcal{L} = \mu$ . We consider how the elements in  $D$  act on the quadruple with  $\mathbf{i} = \emptyset$  in the deformed category  $\mathcal{X}^{\lambda}$  with  $\lambda$  chosen so that  $\sum \lambda_i = \mu$ .

It suffices to check that these elements act linearly independently on  $\mathcal{X}_{\mathbb{K}}^{\lambda}$  for some  $\underline{\lambda}$ ; in the course of the proof we'll modify  $\underline{\lambda}$  as necessary to achieve this. Note the

enormous advantage obtained by having both dominant and anti-dominant weights, as we can add cancelling pairs of these without changing the total sum.

As in [Weba, 4.17], we can compose with the diagram  $\eta_\kappa$  pulling all black strands to the left and  $\dot{\eta}_\kappa$ , its vertical reflection<sup>3</sup>. This will send a non-trivial relation between the diagrams in  $D$  to a non-trivial relation between diagrams where  $\kappa(i) = n$  for all  $i$ .

We can now project this relation to the subspace where we fix the eigenvalue of each dot acting at the top and bottom. The formulas (3.2, 4.5–4.7) defining the functor  $\Xi$  show that this projection is the image under  $\Phi$  of a diagram with an equal or smaller number of crossings, and we can only have equality if we choose eigenvalues so that they coincide at opposite ends of a strand. Now, fix a matching  $D$  such that an associated basis vector appears in our relation, and the corresponding diagram has a maximal number of crossings among those that appear.

We can adjust our weights in the list  $\lambda$  so that they include weights  $\lambda_a$  in bijection with the matching pairs in  $D$  (which correspond to arcs in the diagram), with  $\lambda_a$  dominant if the corresponding arc  $a$  is downward oriented at its lefthand edge and  $\lambda_a$  anti-dominant if the arc  $a$  is upward oriented at its lefthand edge. Now, let us take the projection to the subspace where the eigenvalue of the dot at each end of the arc  $a$  is the variable  $z_a$ . Let  $\mathbf{j}$  and  $\mathbf{j}'$  be the associated sequences in  $\tilde{I}$  at the bottom and top of the diagram.

Note that  $D$  gives the only way of matching the terminals in  $\mathbf{j}$  and  $\mathbf{j}'$  to produce a legal tricolore diagram. Thus all diagrams in our relation that give a different matching from  $D$  project to 0, since there is no matching which has the same eigenvalue at both ends of each strand and fewer crossings than  $D$ .

Therefore, this must be the projection of a relation in  $\mathcal{X}_{\mathbb{K}}^\lambda$  where all terms have the underlying matching  $D$  with some number  $d_a$  of dots on the arc  $a$ , times some monomial  $M$  in the bubbles at the left of the diagram. If we show that no such relation exists, then for each choice of  $d_a$  and  $M$ , the corresponding term must have had coefficient 0 in the original relation. It follows that the original relation must have been trivial.

First, we consider bubbles. Any bubble at the left of the diagram evaluates to a scalar, using the relations (4.1a) and (4.1b). The clockwise bubbles with label  $i$  evaluate to the coefficients of the power series  $\prod_i (u - z_k)^{\lambda_k^i}$  and counterclockwise bubbles to the coefficients of its formal inverse  $\prod_i (u - z_k)^{-\lambda_k^i}$ . By adding new pairs of red and blue strands with labels  $\nu$  and  $-\nu$  for a strictly dominant weight  $\nu$ , we can assure that any finite set of monomials in clockwise bubbles are sent to elements of  $\mathbb{K}$  which are algebraically independent over  $\mathbb{K}$ .

Furthermore, we can explicitly describe the space  $\text{HOM}_{\mathcal{T}}((\underline{\lambda}, \mathbf{j}, \kappa), (\underline{\lambda}, \mathbf{j}', \kappa'))$ ; it has a basis over  $\mathbb{K}$  given diagrams with matching  $D$  and with  $d_a < |\lambda_a^i|$ . Thus, for any finite number of ways of choosing  $d_a$  and  $M$ , we can choose  $\lambda_a'$ 's so the corresponding

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<sup>3</sup>We use  $\eta$  instead of  $\theta$  here since we are pulling left rather than right.



diagrams lie in this basis. Since these diagrams remain linearly independent after acting in  $\mathcal{X}_{\mathbb{K}}^{\underline{\lambda}}$ , they must be linearly independent, so we must have that the coefficient of this diagram in  $\mathbb{K}$  is 0. This in turn supplies a polynomial relation between the values of the clockwise bubbles. We can rule out this possibility by choosing  $\underline{\lambda}$  so that the bubbles which appear in the relation are algebraically independent. Thus, we see that the relation we chose is trivial. This establishes the linear independence of our prospective basis and establishes the result.  $\square$

Since  $\mathcal{U}$  has a natural functor to  $\mathcal{T}$  just not using any red or blue strands, this shows:

**Theorem 4.10** *The 2-category  $\mathcal{U}$  is non-degenerate for any field  $\mathbb{k}$  and choice of  $Q_{*,*}$ . In particular, the Grothendieck group  $K_q(\mathcal{U})$  is isomorphic to  $\dot{\mathbf{U}}$ .*

*Proof.* The spanning set  $B_{i,j,\lambda}$  of Khovanov and Lauda is still linearly independent after applying the functor to  $\mathcal{T}$ , and thus must be linearly independent. This shows the non-degeneracy, and the isomorphism of Grothendieck groups follows from [KL10, 1.2].  $\square$

*Remark 4.11.* If our goal was only to prove Theorem 4.10, we could have avoided some of the difficulties of the proof above: as observed above, we can define a categorical action of  $\mathcal{U}$  on  $\mathcal{X}^{\lambda_1} \otimes \mathcal{X}^{-\lambda_2}$  which categorifies the tensor product of a highest and lowest weight simple, using Theorem 3.4. Note, however, that this requires  $y$  to not be nilpotent, but to have at least two elements in its spectrum. This shows that the map  $\dot{\mathbf{U}} \rightarrow K(\mathcal{U})$  must be injective, since no element of  $\dot{\mathbf{U}}$  kills all highest tensor lowest modules; we can also see that we get the correct inner product, since the Euler form on  $\mathcal{X}^{\lambda_1} \otimes \mathcal{X}^{-\lambda_2}$  matches the tensor product of Shapovalov forms.

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